Measure Theory with Ergodic Horizons
Lecture 29

Cor (Differentiation of all measures). For any loc. finite Borel measure μ on $\mathbb{R}^d$, for a.e. $x \in \mathbb{R}^d$, for any family $\{B_r'(x)\}_{r>0}$ shrinking nicely to x, we have:

$$\lim_{r \to 0} \frac{\mu(B_r'(x))}{\lambda(B_r'(x))} = \frac{d\mu_{\ll}}{d\lambda}(x)$$

where $\mu = \mu_{\ll} + \mu_{\perp}$ is the Lebesgue decomposition wrt λ, i.e. $\mu_{\ll} \ll \lambda$ and $\mu_{\perp} \perp \lambda$.

Lebesgue density.

Lebesgue differentiation theorem for the indicator function of a measurable set $M \subseteq \mathbb{R}^d$ gives:

Lebesgue density theorem. $d_M(x) := \lim_{r \to 0} A_r \mathbb{1}_M = \lim_{r \to 0} \frac{\lambda(M \cap B_r(x))}{\lambda(B_r(x))} = \mathbb{1}_M(x)$ for a.e. $x \in \mathbb{R}^d$.

In fact, the $\{B_r(x)\}_{r>0}$ can be replaced with any family that shrinks nicely to x.

We call d_M the Lebesgue density function (which is defined only on a conull set because $\lim_{r \to 0} A_r \mathbb{1}_M$ may not exist) and we call the set

$$D_M := \{x \in \mathbb{R}^d : d_M = 1\}$$

the Lebesgue density set of M. The theorem above implies $D_M =_\lambda M$, i.e. $D_M \Delta M$ is λ-null. Thus D_M is a canonical element of the $=_\lambda$-equivalence class of M. Indeed, if $M_0 =_\lambda M_1$ then $D_{M_0} = D_{M_1}$. Thus, the map $M \mapsto D_M$ is a selector for the equivalence relation $=_\lambda$ on the set Meas_λ of all λ-measurable sets. We also get:

Strong 99% lemma. For every measurable set $M \subseteq \mathbb{R}^d$ of positive Lebesgue measure, for a.e. $x \in M$ (in fact all $x \in D_M$), for every small enough $r > 0$,

$$\frac{\lambda(M \cap B_r(x))}{\lambda(B_r(x))} \geq 0.99.$$

Examples.

(a) If $U \subseteq \mathbb{R}^d$ is open, then $D_U = U$.

(b) If $B \subseteq \mathbb{R}^d$ is a box, then $D_B = \mathrm{interior}(B)$.

(c) $D_{\mathbb{R}^d \setminus \mathbb{Q}^d} = D_{\mathbb{R}^d} = \mathbb{R}^d$.

(d) If $M \subseteq \mathbb{R}^d$ is λ-null, then $D_M = D_\emptyset = \emptyset$.

(e) $M \mapsto D_M$ is an idempotent, i.e. $D_{D_M} = D_M$ for all $M \in \mathrm{Meas}_\lambda$.

Lebesgue density topology on $\mathbb{R}^d$. Call a λ-measurable set $M \subseteq \mathbb{R}^d$ Lebesgue open if $M \subseteq D_M$. Note that for each M and a null set Z, the set $D_M \setminus Z$ is Lebesgue open, so there are $2^{\mathbb{R}}$-many Lebesgue open sets. It turns out that the Lebesgue open sets form a topology on $\mathbb{R}^d$, called the Lebesgue density topology. This in particular implies that arbitrary unions of Lebesgue open sets are again Lebesgue open, hence λ-measurable!!! This topology is not metrizable and not 2^{nd} ctbl or separable, but it is strong Choquet, in particular the Baire category theorem holds. Also, the Lebesgue meagre sets are exactly the Lebesgue null sets.

Borel measures on $\mathbb{R}$ and the Fundamental Theorem of Calculus (Newton-Leibnitz).

We saw in HW that if a locally finite Borel measure μ on $\mathbb{R}$ had a continuously differentiable distribution $f: \mathbb{R} \to \mathbb{R}$ (i.e. a function f such that $\mu((a,b]) = f(b) - f(a)$) then $\mu \ll \lambda$ and $\frac{d\mu}{d\lambda} = f'$ a.e. Turns out this is true more generally and characterizes those μ which are absolutely continuous wrt λ.

Theorem (Characterization of absolutely continuous measures via their distributions). Let μ be a loc. finite Borel measure on $\mathbb{R}$ and let f be a distribution of μ, i.e. $\mu((a,b]) = f(b) - f(a)$ $\forall\, a \leq b$ (recall that this is equivalent to f being increasing and right continuous). Then f' exists a.e. and $f' \in L^1_{loc}(\mathbb{R}, \lambda)$.

In fact $f' = \frac{d\mu_{\ll}}{d\lambda}$ a.e. where $\mu = \mu_{\ll} + \mu_{\perp}$ is the Lebesgue decomposition of μ wrt λ.
Thus:

(i) $\mu \ll \lambda \iff$ the Fundamental Theorem of Calculus (FTC) holds for f:
$$f(b) - f(a) = \int_a^b f' \, d\lambda$$
for all $a \leq b$.

(ii) $\mu \perp \lambda \iff f' = 0$ a.e.

Proof. For each $x \in \mathbb{R}$, $f'(x) := \lim_{r \to 0} \frac{f(x+r) - f(x)}{r}$, where r can be positive or negative. To show that this limit exists and is equal to $d\mu_{\ll}/d\lambda$, it is enough to prove
$$\lim_{r \to 0^+} \frac{f(x+r) - f(x)}{r} = \frac{d\mu_{\ll}}{d\lambda}(x) \quad \text{and} \quad \lim_{r \to 0^+} \frac{f(x) - f(x-r)}{r} = \frac{d\mu_{\ll}}{d\lambda}(x)$$
for a.e. $x \in \mathbb{R}$. But $f(x+r) - f(x) = \mu((x, x+r])$ and $f(x) - f(x-r) = \mu((x-r, x])$, and the families $\{(x, x+r]\}_{r>0}$ and $\{(x-r, x]\}_{r>0}$ both shrink nicely to x, and $r = \lambda((x, x+r]) = \lambda((x-r, x])$, so by the theorem about differentiation of arbitrary loc. fin. Borel measures, we get that for a.e. $x \in \mathbb{R}^d$,
$$\lim_{r \to 0^+} \frac{f(x+r) - f(x)}{r} = \lim_{r \to 0^+} \frac{\mu((x, x+r])}{\lambda((x, x+r])} = \frac{d\mu_{\ll}}{d\lambda}(x) = \lim_{r \to 0^+} \frac{\mu((x-r, x])}{\lambda((x-r, x])} = \lim_{r \to 0^+} \frac{f(x) - f(x-r)}{r}.$$

For (i), $\mu \ll \lambda \iff \mu = \mu_{\ll} \iff \mu((a,b]) = \int_a^b f' \, d\lambda$ for all $a \leq b$
$$\iff f(b) - f(a) = \int_a^b f' \, d\lambda \text{ for all } a \leq b,$$
where $\Leftarrow$ holds by the uniqueness of Carathéodory extension since the sets

$[a,b]$ generate the Borel sigma-algebra so if μ is equal to the measure λ_f on these sets, then $\mu = \lambda_f$.

For (ii), $\mu \perp \lambda \iff \mu_{\ll} = 0 \iff \frac{d\mu_{\ll}}{d\lambda} = 0$ a.e. $\iff f' = 0$ a.e. $\square$